Lepton flavor model from $\Delta(54)$ symmetry

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ABSTRACT: We present the lepton flavor model with $\Delta(54)$, which appears typically in heterotic string models on the T^2/Z_3 orbifold. Our model reproduces the tri-bimaximal mixing in the parameter region around degenerate neutrino masses or two massless neutrinos. We predict the deviation from the tri-bimaximal mixing by putting the experimental data of neutrino masses in the normal hierarchy of neutrino masses. The upper bound of $\sin^2\theta_{13}$ is 0.01. There is the strong correlation between θ_{23} and θ_{13}. Unless θ_{23} is deviated from the maximal mixing considerably, θ_{13} remains to be tiny.

JHEP04(2009)011

Contents

1 Introduction

It is the important task to find an origin of the observed hierarchies in masses and flavor mixing for quarks and leptons. Neutrino experimental data provide us a valuable clue to find this origin. In particular, recent experiments of the neutrino oscillation go into the new phase of precise determination of mixing angles and mass squared differences [1]. Those indicate the tri-bimaximal mixing for three flavors in the lepton sector [2]. Therefore, it is necessary to find a natural model that leads to this mixing pattern with good accuracy.

The tri-bimaximal mixing for three flavors indicates the specific neutrino mass matrix, in which matrix elements are connected each other. The non-Abelian discrete flavor symmetry is appropriate to explain such a structure of the mass matrix which leads to the tri-bimaximal, because the symmetry provides the definite meaning of generations and connects different generations. Actually, several types of models with various non-Abelian discrete flavor symmetries have been proposed, such as S_3 [3]–[19], D_4 [20]–[24], D_6 [25], Q_4 [26], Q_6 [27], A_4 [28]–[48], T' [49]–[54], S_4 [55]–[60] and $\Delta(27)$ [61]–[65].

Non-Abelian discrete symmetries are symmetries of geometrical solids. Thus, an origin of non-Abelian discrete flavor symmetries may be compact extra dimensions, e.g. string-derived compact spaces. Recently, which types of non-Abelian discrete flavor symmetries can appear in heterotic orbifold models has been studied [66–68]. Simple orbifolds can lead to non-Abelian flavor symmetries such as D_4, $\Delta(54)$ and so on. For example, the

	(l_e, l_μ, l_τ)	(e^c, μ^c, τ^c)	$(N_e^c, N_\mu^c, N_\tau^c)$	$h_{u(d)}$	χ_1	(χ_2, χ_3)	(χ_4, χ_5, χ_6)
$\Delta(54)$	$3_1^{(1)}$	$3_2^{(2)}$	$3_1^{(2)}$	1_1	1_2	2_1	$3_1^{(2)}$

Table 1. Assignments of $\Delta(54)$ representations

$\Delta(54)$ flavor symmetry can appear typically in heterotic string models on factorizable orbifolds including the T^2/Z_3 orbifold. Other string compactifications would lead to different flavor symmetries.

The D_4 flavor model has been already proposed by Grimus and Lavoura [20] and phenomenologically important results have been obtained [22]. The $\Delta(54)$ flavor symmetry would be also interesting, e.g. from the viewpoint that $\Delta(54)$ has triplet irreducible representations [69], while D_4 has only singlets and doublets. Indeed, non-Abelian flavor symmetries, A_4, S_4, and T', include triplet irreducible representations and those are useful to explain the three generations of leptons with their mixing angles and reproduce the tri-bimaximal mixing of flavors. The $\Delta(54)$ flavor symmetry would have similarly interesting aspects. However, the group $\Delta(54)$ is rather unfamiliar compared to other discrete groups used as the flavor symmetry. Its phenomenological applications have not been studied. Thus, our purpose in this paper is to present a lepton flavor model with the $\Delta(54)$ symmetry and study phenomenological implications.

The paper is organized as follows: we present the framework of the lepton flavor model with $\Delta(54)$ in section 2, and discuss the effect of the higher order corrections, in section 3. In section 4, we present the potential analysis to assure the VEVs used in section 2. Numerical results are exhibited in section 5 for neutrino masses and mixing angles. Section 6 is devoted to summary and discussion. In the appendix, we present the character table, the kronecker products and Clebsch Gordan coefficients of $\Delta(54)$.

2 $\Delta(54)$ lepton flavor model

In this section, we present the lepton flavor model with the $\Delta(54)$ flavor symmetry. We propose our model within the framework of supersymmetric models. However, similar non-supersymmetric models could be constructed.

The $\Delta(54)$ group is one of series of $\Delta(6n^2)$ that has been discussed by a few authors [69, 70]. The group $\Delta(54)$ has irreducible representations 1_1, 1_2, 2_1, 2_2, 2_3, 2_4, $3_1^{(1)}$, $3_1^{(2)}$, $3_2^{(1)}$, and $3_2^{(2)}$. It is remarked that there are four triplets and only $3_1^{(1)} \times 3_1^{(2)}$ leads to the trivial singlet. The relevant multiplication rules are summarized in appendix.

Let us present the model of the lepton flavor with the $\Delta(54)$ group. The triplet representations of the group correspond to the three generations of leptons. The left-handed leptons (l_e, l_μ, l_τ), the right-handed charged leptons (e^c, μ^c, τ^c) and the right-handed neutrinos $(N_e^c, N_\mu^c, N_\tau^c)$ are assigned by $3_1^{(1)}$, $3_2^{(2)}$, and $3_1^{(2)}$, respectively. Since $3_1^{(1)} \times 3_1^{(2)}$ makes trivial singlet 1_1, only Dirac neutrino Yukawa couplings are allowed in tree level. On the other hand, charged leptons and the right-handed Majorana neutrinos cannot have mass terms unless new scalars χ_i are introduced in addition to the usual Higgs doublets, h_u and h_d.

These new scalars are supposed to be SU(2) gauge singlets. The gauge singlets χ_1, (χ_2, χ_3) and (χ_4, χ_5, χ_6) are assigned to 1_2, 2_1, and $3_1^{(2)}$ of the $\Delta(54)$ representations, respectively. The particle assignments of $\Delta(54)$ are summarized in table 1. The usual Higgs doublets h_u and h_d are assigned to the trivial singlet 1_1 of $\Delta(54)$. Here, we use the conventional notation that we denote the superfield and its lowest scalar component by the same letter.

In this setup of the particle assignment, let us consider the superpotential of leptons at the leading order in terms of the cut-off scale Λ, which is taken to be the Planck scale. For charged leptons, the superpotential of the Yukawa sector respecting to $\Delta(54)$ symmetry is given as

$$
\begin{aligned}
w_l = {}& y_1^l (e^c l_e + \mu^c l_\mu + \tau^c l_\tau) \chi_1 h_d / \Lambda \\
&+ y_2^l \left[(\omega e^c l_e + \omega^2 \mu^c l_\mu + \tau^c l_\tau) \chi_2 - (e^c l_e + \omega^2 \mu^c l_\mu + \omega \tau^c l_\tau) \chi_3 \right] h_d / \Lambda.
\end{aligned} \tag{2.1}
$$

For the right-handed Majorana neutrinos we can write the superpotential as follows:

$$
\begin{aligned}
w_N = {}& y_1 (N_e^c N_e^c \chi_4 + N_\mu^c N_\mu^c \chi_5 + N_\tau^c N_\tau^c \chi_6) \\
&+ y_2 \left[(N_\mu^c N_\tau^c + N_\tau^c N_\mu^c) \chi_4 + (N_e^c N_\tau^c + N_\tau^c N_e^c) \chi_5 + (N_e^c N_\mu^c + N_\mu^c N_e^c) \chi_6 \right].
\end{aligned} \tag{2.2}
$$

The superpotential for the Dirac neutrinos has tree level contributions as

$$
w_D = y_D \, (N_e^c l_e + N_\mu^c l_\mu + N_\tau^c l_\tau) h_u \, . \tag{2.3}
$$

We assume that the scalar fields, $h_{u,d}$ and χ_i, develop their vacuum expectation values (VEVs) as follows:

$$
\begin{aligned}
&\langle h_u \rangle = v_u, && \langle h_d \rangle = v_d, \\
&\langle \chi_1 \rangle = u_1, && \langle (\chi_2, \chi_3) \rangle = (u_2, u_3), && \langle (\chi_4, \chi_5, \chi_6) \rangle = (u_4, u_5, u_6).
\end{aligned} \tag{2.4}
$$

Then, we obtain the diagonal mass matrix for charged leptons

$$
M_l = y_1^l v_d \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix} + y_2^l v_d \begin{pmatrix} \omega\alpha_2 - \alpha_3 & 0 & 0 \\ 0 & \omega^2\alpha_2 - \omega^2\alpha_3 & 0 \\ 0 & 0 & \alpha_2 - \omega\alpha_3 \end{pmatrix}, \tag{2.5}
$$

while the right-handed Majorana mass matrix is given as

$$
M_N = y_1 \Lambda \begin{pmatrix} \alpha_4 & 0 & 0 \\ 0 & \alpha_5 & 0 \\ 0 & 0 & \alpha_6 \end{pmatrix} + y_2 \Lambda \begin{pmatrix} 0 & \alpha_6 & \alpha_5 \\ \alpha_6 & 0 & \alpha_4 \\ \alpha_5 & \alpha_4 & 0 \end{pmatrix}, \tag{2.6}
$$

and the Dirac mass matrix of neutrinos is

$$
M_D = y_D v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.7}
$$

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where we denote $\alpha_i = u_i/\Lambda$ $(i = 1-6)$. By using the seesaw mechanism $M_\nu = M_D^T M_N^{-1} M_D$, the neutrino mass matrix can be written as

$$M_\nu = \frac{y_D^2 v_u^2}{\Lambda d} \begin{pmatrix} y_1^2 \alpha_5 \alpha_6 - y_2^2 \alpha_4^2 & -y_1 y_2 \alpha_6^2 + y_2^2 \alpha_4 \alpha_5 & -y_1 y_2 \alpha_5^2 + y_2^2 \alpha_4 \alpha_6 \\ -y_1 y_2 \alpha_6^2 + y_2^2 \alpha_4 \alpha_5 & y_1^2 \alpha_4 \alpha_6 - y_2^2 \alpha_5^2 & -y_1 y_2 \alpha_4^2 + y_2^2 \alpha_5 \alpha_6 \\ -y_1 y_2 \alpha_5^2 + y_2^2 \alpha_4 \alpha_6 & -y_1 y_2 \alpha_4^2 + y_2^2 \alpha_5 \alpha_6 & y_1^2 \alpha_4 \alpha_5 - y_2^2 \alpha_6^2 \end{pmatrix},$$

$$d = y_1^3 \alpha_4 \alpha_5 \alpha_6 - y_1 y_2^2 \alpha_4^3 - y_1 y_2^2 \alpha_5^3 - y_1 y_2^2 \alpha_6^3 + 2 y_2^3 \alpha_4 \alpha_5 \alpha_6. \tag{2.8}$$

Since the charged leptons mass matrix is diagonal one, we can simply get the mass eigenvalues as

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = v_d \begin{pmatrix} 1 & \omega & -1 \\ 1 & \omega^2 & -\omega^2 \\ 1 & 1 & -\omega \end{pmatrix} \begin{pmatrix} y_1^l \alpha_1 \\ y_2^l \alpha_2 \\ y_2^l \alpha_3 \end{pmatrix}. \tag{2.9}$$

In order to estimate magnitudes of α_1, α_2 and α_3, we rewrite as

$$\begin{pmatrix} y_1^l \alpha_1 \\ y_2^l \alpha_2 \\ y_2^l \alpha_3 \end{pmatrix} = \frac{1}{3 v_d} \begin{pmatrix} 1 & 1 & 1 \\ -\omega - 1 & \omega & 1 \\ -1 & -\omega & \omega + 1 \end{pmatrix} \begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix}, \tag{2.10}$$

which gives the relation of $|y_2^l \alpha_2| = |y_2^l \alpha_3|$. Inserting the experimental values of the charged lepton masses and $v_d \simeq 55\text{GeV}$, which is given by taking $\tan\beta = 3$, we obtain numerical results

$$\begin{pmatrix} y_1^l \alpha_1 \\ y_2^l \alpha_2 \\ y_2^l \alpha_3 \end{pmatrix} = \begin{pmatrix} 1.14 \times 10^{-2} \\ 1.05 \times 10^{-2} e^{0.016 i\pi} \\ 1.05 \times 10^{-2} e^{0.32 i\pi} \end{pmatrix}. \tag{2.11}$$

Thus, it is found that $\alpha_i (i = 1, 2, 3)$ are order of $\mathcal{O}(10^{-2})$ if the Yukawa couplings are order one.

In our model, the lepton mixing comes from the structure of the neutrino mass matrix of eq. (2.8). In order to reproduce the maximal mixing between ν_μ and ν_τ, we take $\alpha_5 = \alpha_6$, and then we have

$$M_\nu = \frac{y_D^2 v_u^2}{\Lambda d} \begin{pmatrix} y_1^2 \alpha_5^2 - y_2^2 \alpha_4^2 & -y_1 y_2 \alpha_5^2 + y_2^2 \alpha_4 \alpha_5 & -y_1 y_2 \alpha_5^2 + y_2^2 \alpha_4 \alpha_5 \\ -y_1 y_2 \alpha_5^2 + y_2^2 \alpha_4 \alpha_5 & y_1^2 \alpha_4 \alpha_5 - y_2^2 \alpha_5^2 & -y_1 y_2 \alpha_4^2 + y_2^2 \alpha_5^2 \\ -y_1 y_2 \alpha_5^2 + y_2^2 \alpha_4 \alpha_5 & -y_1 y_2 \alpha_4^2 + y_2^2 \alpha_5^2 & y_1^2 \alpha_4 \alpha_5 - y_2^2 \alpha_5^2 \end{pmatrix}. \tag{2.12}$$

The tri-bimaximal mixing is realized by the condition of $M_\nu(1,1) + M_\nu(1,2) = M_\nu(2,2) + M_\nu(2,3)$ in eq. (2.12), which turns to

$$(y_1 - y_2)(\alpha_4 - \alpha_5)(y_1 \alpha_5 - y_2 \alpha_4) = 0. \tag{2.13}$$

Therefore, we have three cases realizing the tri-bimaximal mixing in eq. (2.12) as

$$y_1 = y_2, \qquad \alpha_4 = \alpha_5, \qquad y_1 \alpha_5 = y_2 \alpha_4. \tag{2.14}$$

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Let us investigate the neutrino mass spectrum in these cases. In general the neutrino mass matrix with the tri-bimaximal mixing is expressed as

$$M_\nu = \frac{m_1 + m_3}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2 - m_1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_1 - m_3}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.15}$$

Actually, the neutrino mass matrix of eq. (2.12) is decomposed under the condition in eq. (2.14) as follows. In the case of $\alpha_4 = \alpha_5$, the neutrino mass matrix is expressed as

$$M_\nu = \frac{y_D^2 v_u^2 \alpha_4^2 (y_1 - y_2)}{\Lambda d} \left[(y_1 + 2y_2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - y_2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right]. \tag{2.16}$$

Therefore, it is found that neutrino masses are given as

$$\begin{aligned} \frac{m_1 + m_3}{2} &= \frac{y_D^2 v_u^2 \alpha_4^2 (y_1 - y_2)}{\Lambda d}(y_1 + 2y_2), \\ \frac{m_2 - m_1}{3} &= -\frac{y_D^2 v_u^2 \alpha_4^2}{\Lambda d}(y_1 - y_2)y_2, \\ m_1 - m_3 &= 0. \end{aligned} \tag{2.17}$$

In the case of $y_1 = y_2$, the mass matrix is decomposed as

$$M_\nu = \frac{y_D^2 y_1^2 v^2 (\alpha_4 - \alpha_5)}{\Lambda d} \left[\alpha_5 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - (\alpha_4 + 2\alpha_5) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right], \tag{2.18}$$

and we have

$$\begin{aligned} m_1 + m_3 &= 0, \\ \frac{m_2 - m_1}{3} &= \frac{y_D^2 y_1^2 v_u^2 (\alpha_4 - \alpha_5)}{\Lambda d}\alpha_5, \\ \frac{m_1 - m_3}{2} &= -\frac{y_D^2 y_1^2 v_u^2 (\alpha_4 - \alpha_5)}{\Lambda d}(\alpha_4 + 2\alpha_5). \end{aligned} \tag{2.19}$$

In the last case of $y_1 \alpha_5 = y_2 \alpha_4$, we have

$$M_\nu = \frac{y_D^2 v_u^2}{\Lambda d} y_1^2 \alpha_4 \alpha_5 \left(1 - \frac{\alpha_5^3}{\alpha_4^3}\right) \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right]. \tag{2.20}$$

Then, we obtain

$$\begin{aligned} m_3 &= \frac{2y_D^2 v_u^2}{\Lambda d} y_1^2 \alpha_4 \alpha_5 \left(1 - \frac{\alpha_5^3}{\alpha_4^3}\right), \\ m_2 &= m_1 = 0. \end{aligned} \tag{2.21}$$

Thus, the tri-bimaximal mixing is not realized for arbitrary neutrino masses m_1, m_2 and m_3 in our model. In both conditions of $y_1 = y_2$ and $\alpha_4 = \alpha_5$, we have $|m_1| = |m_3|$,

which leads to quasi-degenerate neutrino masses due to the condition of $\Delta m_{\rm atm}^2 \gg \Delta m_{\rm sol}^2$. Therefore, we do not discuss these cases in this paper because we need fine-tuning of parameters in order to be consistent with the experimental data of the neutrino oscillations [1].

In the case of $y_1\alpha_5 = y_2\alpha_4$, the neutrino mass matrix turns to be

$$M_\nu = \frac{y_D^2 y_1^2 v_u^2}{\Lambda d} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_4\alpha_5 - \alpha_5^4/\alpha_4^2 & -\alpha_4\alpha_5 + \alpha_5^4/\alpha_4^2 \\ 0 & -\alpha_4\alpha_5 + \alpha_5^4/\alpha_4^2 & \alpha_4\alpha_5 - \alpha_5^4/\alpha_4^2 \end{pmatrix}. \quad (2.22)$$

This neutrino matrix is a prototype which leads to the tri-bimaximal mixing with the mass hierarchy $m_3 \gg m_2 \geq m_1$, then we expect that realistic mass matrix is obtained near the condition $y_1\alpha_5 = y_2\alpha_4$.

Let us discuss the detail of the mass matrix (2.12). After rotating $\theta_{23} = 45°$, we get

$$\frac{y_D^2 v_u^2}{\Lambda d} \begin{pmatrix} y_1^2\alpha_5^2 - y_2^2\alpha_4^2 & \sqrt{2}(-y_1 y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5) & 0 \\ \sqrt{2}(-y_1 y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5) & y_1^2\alpha_4\alpha_5 - y_1 y_2\alpha_4^2 & 0 \\ 0 & 0 & y_1^2\alpha_4\alpha_5 + y_1 y_2\alpha_4^2 - 2y_2^2\alpha_5^2 \end{pmatrix}, \quad (2.23)$$

which leads $\theta_{13} = 0$ and

$$\theta_{12} = \frac{1}{2}\arctan\frac{2\sqrt{2}y_2\alpha_5}{y_1\alpha_5 + y_2\alpha_4 - y_1\alpha_4} \qquad (y_2\alpha_4 \neq y_1\alpha_5). \quad (2.24)$$

Neutrino masses are given as

$$\begin{aligned} m_1 &= \frac{y_D^2 v_u^2}{\Lambda d}[y_1^2\alpha_5^2 - y_2^2\alpha_4^2 - \sqrt{2}(-y_1 y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5)\tan\theta_{12}], \\ m_2 &= \frac{y_D^2 v_u^2}{\Lambda d}[y_1^2\alpha_4\alpha_5 - y_1 y_2\alpha_4^2 + \sqrt{2}(-y_1 y_2\alpha_5^2 + y_2^2\alpha_4\alpha_5)\tan\theta_{12}], \\ m_3 &= \frac{y_D^2 v_u^2}{\Lambda d}[y_1^2\alpha_4\alpha_5 + y_1 y_2\alpha_4^2 - 2y_2^2\alpha_5^2], \end{aligned} \quad (2.25)$$

which are reconciled with the normal hierarchy of neutrino masses in the case of $y_1\alpha_5 \simeq y_2\alpha_4$.

Let us estimate magnitudes of $\alpha_i(i = 4, 5, 6)$ by using eq. (2.25). Suppose $\tilde{\alpha} = \alpha_4 \simeq \alpha_5 = \alpha_6$. If we take all Yukawa couplings to be order one, eq. (2.25) turns to be $v_u^2 = \Lambda\tilde{\alpha}m_3$ because of $d \sim \tilde{\alpha}^3$. Putting $v_u \simeq 165\mathrm{GeV}$ ($\tan\beta = 3$), $m_3 \simeq \sqrt{\Delta m_{\rm atm}^2} \simeq 0.05\mathrm{eV}$, and $\Lambda = 2.43 \times 10^{18}\mathrm{GeV}$, we obtain $\tilde{\alpha} = \mathcal{O}(10^{-4} - 10^{-3})$. Thus, values of $\alpha_i(i = 4, 5, 6)$ are enough suppressed to discuss perturbative series of higher mass operators.

3 Higher order corrections

Let us consider higher order contributions to mass matrices. There are six expansion parameters α_i, all of which are expected to be small.

Since products $3_1^{(1)} \times 3_2^{(2)} \times 2_1 \times 1_2$ and $3_1^{(1)} \times 3_2^{(2)} \times 2_1 \times 2_1$ give the $\Delta(54)$ invariant in the charged lepton sector, the superpotential of next leading order is written as

$$\begin{aligned} \delta w_l = y_3^l[(\omega e^c l_e + \omega^2\mu^c l_\mu + \tau^c l_\tau)\chi_2 + (e^c l_e + \omega^2\mu^c l_\mu + \omega\tau^c l_\tau)\chi_3]\chi_1 h_d/\Lambda^2 \\ + y_4^l((e^c l_e + \omega^2\mu^c l_\mu + \omega\tau^c l_\tau)\chi_2^2 - (\omega e^c l_e + \omega^2\mu^c l_\mu + \tau^c l_\tau)\chi_3^2)h_d/\Lambda^2. \quad (3.1) \end{aligned}$$

For the right-handed Majorana neutrinos, the $\Delta(54)$ invariant product $3_1^{(2)} \times 3_1^{(2)} \times 3_1^{(2)} \times 2_1$ gives

$$
\begin{aligned}
\delta w_N = {} & y_3[(\omega N_e^c N_e^c \chi_4 + \omega^2 N_\mu^c N_\mu^c \chi_5 + N_\tau^c N_\tau^c \chi_6)\chi_2 \\
& + (N_e^c N_e^c \chi_4 + \omega^2 N_\mu^c N_\mu^c \chi_5 + \omega N_\tau^c N_\tau^c \chi_6)\chi_3]/\Lambda \\
& + y_4 \left[\{\omega(N_\mu^c N_\tau^c + N_\tau^c N_\mu^c)\chi_4 + \omega^2(N_e^c N_\tau^c + N_\tau^c N_e^c)\chi_5 + (N_e^c N_\mu^c + \bar{N}_\mu^c N_e^c)\chi_6\}\chi_2 \right. \\
& + \{(N_\mu^c N_\tau^c + N_\tau^c N_\mu^c)\chi_4 + \omega^2(N_e^c l_\tau^c + N_\tau^c N_e^c)\chi_5 \\
& \left. + \omega(N_e^c N_\mu + N_\mu^c N_e^c)\chi_6\}\chi_3 \right] /\Lambda.
\end{aligned}
\tag{3.2}
$$

The product $3_1^{(1)} \times 3_1^{(2)} \times 2_1$ gives a $\Delta(54)$ invariant in the Dirac neutrino sector as

$$
\delta w_D = y_2^D[(\omega N_e^c l_e + \omega^2 N_\mu^c l_\mu + N_\tau^c l_\tau)\chi_2 + (N_e^c l_e + \omega^2 N_\mu^c l_\mu + \omega N_\tau^c l_\tau)\chi_3]h_u/\Lambda. \tag{3.3}
$$

These correction terms of the superpotential in eqs. (3.1), (3.2), (3.3) give corrections of mass matrices

$$
\begin{aligned}
\delta M_l = {} & y_3^l v_d \alpha_1 \begin{pmatrix} \omega\alpha_2 + \alpha_3 & 0 & 0 \\ 0 & \omega^2(\alpha_2 + \alpha_3) & 0 \\ 0 & 0 & \alpha_2 + \omega\alpha_3 \end{pmatrix} \\
& + y_4^l v_d \begin{pmatrix} \alpha_2^2 - \omega\alpha_3^2 & 0 & 0 \\ 0 & \omega^2(\alpha_2^2 - \alpha_3^2) & 0 \\ 0 & 0 & \omega\alpha_2^2 - \alpha_3^2 \end{pmatrix},
\end{aligned}
\tag{3.4}
$$

for charged leptons,

$$
\begin{aligned}
\delta M_N = {} & y_3\Lambda \begin{pmatrix} (\omega\alpha_2 + \alpha_3)\alpha_4 & 0 & 0 \\ 0 & \omega^2(\alpha_2 + \alpha_3)\alpha_5 & 0 \\ 0 & 0 & (\alpha_2 + \omega\alpha_3)\alpha_6 \end{pmatrix} \\
& + y_4\Lambda \begin{pmatrix} 0 & (\alpha_2 + \omega\alpha_3)\alpha_6 & \omega^2(\alpha_2 + \alpha_3)\alpha_5 \\ (\alpha_2 + \omega\alpha_3)\alpha_6 & 0 & (\omega\alpha_2 + \alpha_3)\alpha_4 \\ \omega^2(\alpha_2 + \alpha_3)\alpha_5 & (\omega\alpha_2 + \alpha_3)\alpha_4 & 0 \end{pmatrix},
\end{aligned}
\tag{3.5}
$$

for right-handed Majorana neutrinos, and

$$
\delta M_D = y_2^D v_u \begin{pmatrix} \omega\alpha_2 + \alpha_3 & 0 & 0 \\ 0 & \omega^2\alpha_2 + \omega^2\alpha_3 & 0 \\ 0 & 0 & \alpha_2 + \omega\alpha_3 \end{pmatrix}, \tag{3.6}
$$

for Dirac neutrinos. It is noticed that the corrections of the mass matrices do not change the zero textures in the leading mass matrices of eqs. (2.5), (2.6), (2.7).

Since the magnitudes of $\alpha_i(i = 1, 2, 3)$ are of $\mathcal{O}(10^{-2})$ as seen in eq. (2.11), mass matrix corrections δM_l and δM_D in eqs. (3.4) and (3.6) are suppressed enough. On the other hand, the magnitudes of $\alpha_i(i = 4, 5, 6)$ are $\mathcal{O}(10^{-4} - 10^{-3})$ as discussed in the previous section. Therefore, the correction δM_N in eq. (3.5) is also suppressed enough. In conclusion, we can neglect the higher order contribution in our numerical study of neutrino masses and mixing angles.

4 Vacuum alignment

We analyze the scalar potential to find out the vacuum alignment.[1] The scalar potential becomes rather simple in the $\Delta(54)$ symmetry. Especially, the supersymmetry is important to see the vacuum alignment.

The $\Delta(54)$ invariant superpotential is given as

$$
\begin{aligned}
w = {} & \mu_1 \chi_1^2 + \mu_2 \chi_2 \chi_3 \\
& + \eta_2(\chi_2^3 + \chi_3^3) + \eta_3(\chi_4^3 + \chi_5^3 + \chi_6^3) + \eta_3' \chi_4 \chi_5 \chi_6 \\
& + \frac{\lambda_1}{\Lambda} \chi_1^4 + \frac{\lambda_2}{\Lambda} \chi_2^2 \chi_3^2 + \frac{\lambda_3}{\Lambda} \chi_1^2 \chi_2 \chi_3 + \frac{\lambda_4}{\Lambda} \chi_1(\chi_2^3 - \chi_3^3) \\
& + \frac{\lambda_6}{\Lambda} \left[\chi_2(\omega \chi_4^3 + \omega^2 \chi_5^3 + \chi_6^3) + \chi_3(\chi_4^3 + \omega^2 \chi_5^3 + \omega \chi_6^3) \right] ,
\end{aligned}
\tag{4.1}
$$

which leads to the scalar potential

$$
\begin{aligned}
V = {} & \left| 2\mu_1 \chi_1 + 4\frac{\lambda_1}{\Lambda} \chi_1^3 + 2\frac{\lambda_3}{\Lambda} \chi_1 \chi_2 \chi_3 + \frac{\lambda_4}{\Lambda}(\chi_2^3 - \chi_3^3) \right|^2 \\
& + \left| \mu_2 \chi_3 + 3\eta_2 \chi_2^2 + 2\frac{\lambda_2}{\Lambda} \chi_2 \chi_3^2 + \frac{\lambda_3}{\Lambda} \chi_1^2 \chi_3 + 3\frac{\lambda_4}{\Lambda} \chi_1 \chi_2^2 + \frac{\lambda_6}{\Lambda}(\omega \chi_4^3 + \omega^2 \chi_5^3 + \chi_6^3) \right|^2 \\
& + \left| \mu_2 \chi_2 + 3\eta_2 \chi_3^2 + 2\frac{\lambda_2}{\Lambda} \chi_2^2 \chi_3 + \frac{\lambda_3}{\Lambda} \chi_1^2 \chi_2 - 3\frac{\lambda_4}{\Lambda} \chi_1 \chi_3^2 + \frac{\lambda_6}{\Lambda}(\chi_4^3 + \omega^2 \chi_5^3 + \omega \chi_6^3) \right|^2 \\
& + \left| 3\eta_3 \chi_4^2 + \eta_3' \chi_5 \chi_6 + 3\frac{\lambda_6}{\Lambda}(\omega \chi_2 + \chi_3)\chi_4^2 \right|^2 + \left| 3\eta_3 \chi_5^2 + \eta_3' \chi_4 \chi_6 + 3\frac{\lambda_6}{\Lambda} \omega^2(\chi_2 + \chi_3)\chi_5^2 \right|^2 \\
& + \left| 3\eta_3 \chi_6^2 + \eta_3' \chi_4 \chi_5 + 3\frac{\lambda_6}{\Lambda}(\chi_2 + \omega \chi_3)\chi_6^2 \right|^2 .
\end{aligned}
\tag{4.2}
$$

VEVs of χ_i must be much larger than the weak scale. We assume that their VEVs are determined with neglecting supersymmetry breaking terms, i.e. $V_{\min} = 0$. Then, the conditions of the potential minimum, $V_{\min} = 0$ are written as

$$
\begin{aligned}
2\mu_1 \chi_1 + 4\frac{\lambda_1}{\Lambda} \chi_1^3 + 2\frac{\lambda_3}{\Lambda} \chi_1 \chi_2 \chi_3 + \frac{\lambda_4}{\Lambda}(\chi_2^3 - \chi_3^3) &= 0, \\
\mu_2 \chi_3 + 3\eta_2 \chi_2^2 + 2\frac{\lambda_2}{\Lambda} \chi_2 \chi_3^2 + \frac{\lambda_3}{\Lambda} \chi_1^2 \chi_3 + 3\frac{\lambda_4}{\Lambda} \chi_1 \chi_2^2 + \frac{\lambda_6}{\Lambda}(\omega \chi_4^3 + \omega^2 \chi_5^3 + \chi_6^3) &= 0, \\
\mu_2 \chi_2 + 3\eta_2 \chi_3^2 + 2\frac{\lambda_2}{\Lambda} \chi_2^2 \chi_3 + \frac{\lambda_3}{\Lambda} \chi_1^2 \chi_2 - 3\frac{\lambda_4}{\Lambda} \chi_1 \chi_3^2 + \frac{\lambda_6}{\Lambda}(\chi_4^3 + \omega^2 \chi_5^3 + \omega \chi_6^3) &= 0, \\
3\eta_3 \chi_4^2 + \eta_3' \chi_5 \chi_6 + 3\frac{\lambda_6}{\Lambda}(\omega \chi_2 + \chi_3)\chi_4^2 &= 0, \\
3\eta_3 \chi_5^2 + \eta_3' \chi_4 \chi_6 + 3\frac{\lambda_6}{\Lambda} \omega^2(\chi_2 + \chi_3)\chi_5^2 &= 0, \\
3\eta_3 \chi_6^2 + \eta_3' \chi_4 \chi_5 + 3\frac{\lambda_6}{\Lambda}(\chi_2 + \omega \chi_3)\chi_6^2 &= 0.
\end{aligned}
\tag{4.3}
$$

A solution of the last three equations is

$$
\chi_4 = \chi_5 = \chi_6 \qquad \text{with} \qquad 3\eta_3 + \eta_3' = 0 ,
\tag{4.4}
$$

---

[1]Instead of analyzing the potential minimum, the vacuum alignment could be realized by imposing boundary conditions of χ_i in extra dimensions [71, 72].
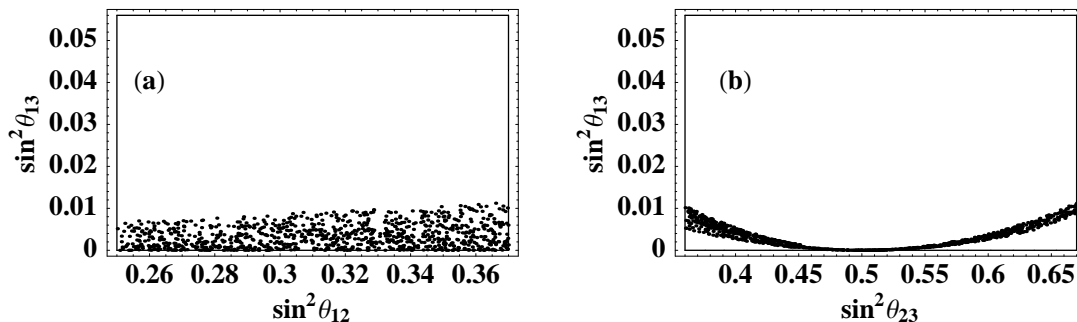
Figure 1. Prediction of the upper bound of $\sin^2\theta_{13}$ on (a) $\sin^2\theta_{12}-\sin^2\theta_{13}$ and (b) $\sin^2\theta_{23}-\sin^2\theta_{13}$ planes.
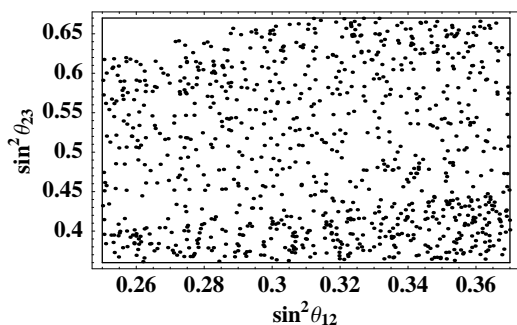

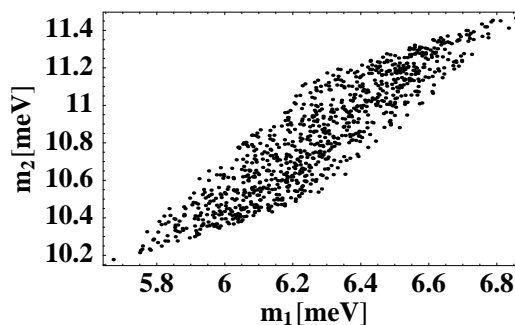**Figure 2**. Allowed region on $\sin^2\theta_{12}-\sin^2\theta_{23}$ plane.

Figure 3. The allowed mass region on the m_1-m_2 plane.

where the higher dimensional operators proportional to λ_6 are neglected. If we include the λ_6 terms, the relation $\alpha_4 = \alpha_5 = \alpha_6$ is deviated in order of $\mathcal{O}(\alpha_i^2)$. Therefore, we take randomly $\alpha_i(i = 4, 5, 6)$ around $\alpha_4 = \alpha_5 = \alpha_6$ in our numerical analysis.

5   Numerical result

We show our numerical analysis of neutrino masses and mixing angles in the normal mass hierarchy. Neglecting higher order corrections of mass matrices in section 3, we obtain the allowed region of parameters and predictions of neutrino masses and mixing angles. Here, we neglect the renomarization effect of the neutrino mass matrix because we suppose the normal hierarchy of neutrino masses and take $\tan\beta = 3$.

Input data of masses and mixing angles are taken in the region of 3σ of the experimental data [1]:

$$\Delta m^2_{\text{atm}} = (2.07 \sim 2.75) \times 10^{-3} \text{eV}^2\,, \quad \Delta m^2_{\text{sol}} = (7.05 \sim 8.34) \times 10^{-5} \text{eV}^2\,,$$
$$\sin^2\theta_{\text{atm}} = 0.36 \sim 0.67\,, \qquad \sin^2\theta_{\text{sol}} = 0.25 \sim 0.37\,, \qquad \sin^2\theta_{\text{reactor}} \le 0.056\,, \quad (5.1)$$

and $\Lambda = 2.43 \times 10^{18}$GeV is taken. We fix $y_D = y_1 = 1$ as a convention, and vary y_2/y_1. The change of y_D and y_1 is absorbed into the change of $\alpha_i(i = 4, 5, 6)$. If we take a smaller value of y_1, values of α_i scale up. On the other hand, if we take a smaller value of y_D, the
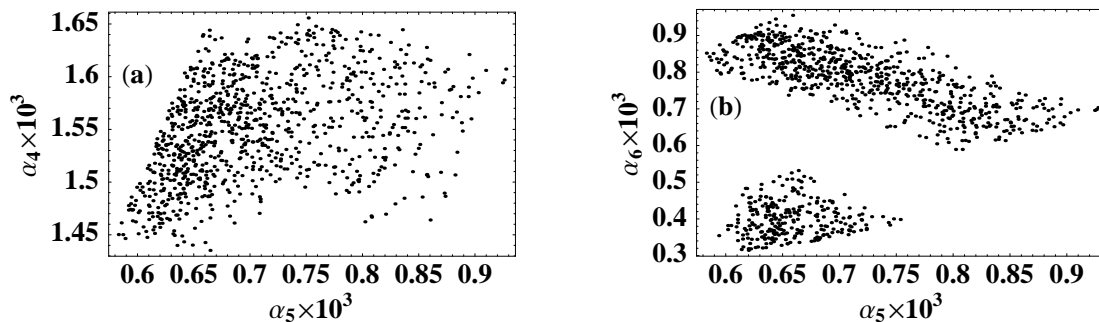
Figure 4. Allowd regions on (a) $\alpha_5 - \alpha_4$ and (b) $\alpha_5 - \alpha_6$ planes.
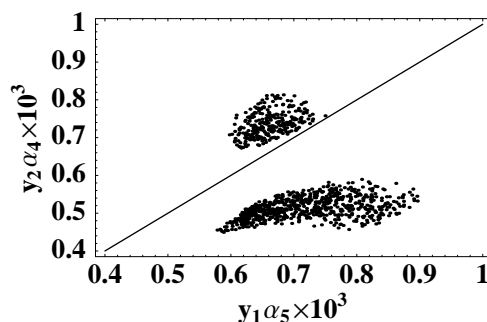

Figure 5. The allowed region on $y_1\alpha_5 - y_2\alpha_4$ plane. The solid line denote $y_1\alpha_5 = y_2\alpha_4$ one.

magnitude of α_i scale down. As expected in the discussion of section 2, the experimentally allowed values are reproduced around $\alpha_4 = \alpha_5 = \alpha_6$.

We can predict the deviation from the tri-bimaximal mixing. The remarkable prediction is given in the magnitude of $\sin^2\theta_{13}$. In figures 1 (a) and (b), we plot the allowed region of mixing angles in planes of $\sin^2\theta_{12}$-$\sin^2\theta_{13}$ and $\sin^2\theta_{23}$-$\sin^2\theta_{13}$, respectively. It is found that the upper bound of $\sin^2\theta_{13}$ is 0.01. It is also found the strong correlation between $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. Unless θ_{23} is deviated from the maximal mixing considerably, θ_{13} remains to be tiny. This is a testable relation in this model.

The allowed region on the $\sin^2\theta_{12}$-$\sin^2\theta_{23}$ plane is presented in figure 2. There is no correlation between $\sin^2\theta_{23}$ and $\sin^2\theta_{12}$ as well as between $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$.

Let us discuss the first and second neutrino masses on the $m_1 - m_2$ plane in figure 3. We find the lightest neutrino mass m_1 in the very narrow region of $m_1 \simeq (6-7) \times 10^{-3}$eV in our model.

In figure 4, we present allowed regions of parameters of α_4, α_5 and α_6, which give the neutrino masses and mixing angles consistent with the experimental data. It is found $\alpha_4 \sim \alpha_5 \sim \alpha_6 \sim \mathcal{O}(10^{-3})$, which can be realized in the potential analysis of the previous section. Since the magnitude of α_i is found to be $\mathcal{O}(10^{-3})$ as expected in the section 2, the neglect of the higher order corrections on the mass matrices are guaranteed.

At last, we discuss about the relation of $y_1\alpha_5 \simeq y_2\alpha_4$, which is expected in our analysis

as discussed in eq. (2.22) of section 2. This relation is well satisfied in our numerical result, which is shown on $y_1\alpha_5 - y_2\alpha_4$ plane in figure 5. By taking account of both results in figure 4(a) and figure 5, we have found that the ratio y_2/y_1 is constrained around $0.3 - 0.5$. Thus, Yukawa couplings y_1 and y_2 are of the same order.

6 Summary and discussion

We have presented the flavor model for the lepton mass matrices by using the discrete symmetry $\Delta(54)$, which could be originated from the string orbifold. The left-handed leptons, the right-handed charged leptons and the right-handed neutrinos are assigned by $3_1^{(1)}$, $3_2^{(2)}$, and $3_1^{(2)}$, respectively. We introduce gauge singlets χ_1, (χ_2, χ_3) and (χ_4, χ_5, χ_6), which are assigned to be 1_2, 2_1, and $3_1^{(2)}$ of the $\Delta(54)$ representations, respectively. The $\Delta(54)$ flavor symmetry can appear in heterotic string models on factorizable orbifolds including the T^2/Z_3 orbifold [67]. In these string models only singlets and triplets appear as fundamental modes, but doublets do not appear as fundamental modes. The doublet plays an role in our model, and such doublet could appear, e.g. as composite modes of triplets.

As discussed in eqs. (2.15)–(2.21), the tri-bimaximal mixing is not realized for arbitrary neutrino masses in our model. Parameters are adapted to get neutrino masses consistent with observed values of $\Delta m_{\rm atm}^2$ and $\Delta m_{\rm sol}^2$. Then, the deviation from the tri-bimaximal mixing is estimated. Therefore, our approach does not predict the tri-bimaximal mixing, but constrain the neutrino mass matrix by putting $\theta_{23} \simeq \pi/4$ by hand.

It is useful to give a following comment as to $\Delta(27)$ flavor symmetry. Our mass matrix gives the same result in the $\Delta(27)$ flavor symmetry [64], where the type II seesaw is used. Our neutrino mass matrix is given as $M_\nu \propto M_N^{-1}$, where M_N is the just same as the neutrino mass matrix M_ν in the $\Delta(27)$ flavor symmetry [64]. Therefore, if the type I seesaw is used in the $\Delta(27)$ flavor symmetry, the same neutrino mass matrix can be obtained.

The model reproduces the almost tri-bimaximal mixing in the parameter region around two vanishing neutrino masses. We have predicted the deviation from the tri-bimaximal mixing by input of the experimental data of $\Delta m_{\rm atm}^2$ and $\Delta m_{\rm sol}^2$ in the case of normal hierarchy of neutrino masses. We have found that the upper bound of $\sin^2\theta_{13}$ is 0.01. There is the strong correlation between $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. Unless θ_{23} is deviated from the maximal mixing considerably, θ_{13} remains to be tiny. Therefore, the model is testable in the future neutrino experiments.

irrep	1a	6a	6b	3a	3b	3c	2a	3d	3e	3f
	(1)	(9)	(9)	(6)	(6)	(6)	(9)	(6)	(1)	(1)
$\mathbf{1}_1$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_2$	1	-1	-1	1	1	1	-1	1	1	1
$\mathbf{2}_1$	2	0	0	2	-1	-1	0	-1	2	2
$\mathbf{2}_2$	2	0	0	-1	-1	-1	0	2	2	2
$\mathbf{2}_3$	2	0	0	-1	-1	2	0	-1	2	2
$\mathbf{2}_4$	2	0	0	-1	2	-1	0	-1	2	2
$\mathbf{3}_2^{(1)}$	3	$-\bar{\omega}$	$-\omega$	0	0	0	-1	0	$3\bar{\omega}$	3ω
$\mathbf{3}_2^{(2)}$	3	$-\omega$	$-\bar{\omega}$	0	0	0	-1	0	3ω	$3\bar{\omega}$
$\mathbf{3}_1^{(2)}$	3	ω	$\bar{\omega}$	0	0	0	1	0	3ω	$3\bar{\omega}$
$\mathbf{3}_1^{(1)}$	3	$\bar{\omega}$	ω	0	0	0	1	0	$3\bar{\omega}$	3ω

Table 2. Character table of the group $\Delta(54)$.

A Character table of $\Delta(54)$

Group-theoretical aspects of $\Delta(54)$ can be found in ref. [69], in which $\Delta(6n^2)$ is investigated. $\Delta(54)$ is a discrete subgroup of SU(3), i.e. the group $\Delta(6n^2)$ (with $n = 3$) and it has order 54. The generators of $\Delta(54)$ are given by the set

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c' = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{(A.1)}$$

It has four three-dimensional irreducible representations $\mathbf{3}_1^{(1)}$, $\mathbf{3}_1^{(2)}$, $\mathbf{3}_2^{(1)}$, $\mathbf{3}'^{(2)}_2$, four two-dimensional ones $\mathbf{2}_1$, $\mathbf{2}_2$, $\mathbf{2}_3$, $\mathbf{2}_4$, and two one-dimensional ones $\mathbf{1}_1$, $\mathbf{1}_2$. Generators of three-dimensional representations are mainly divided into two types. For $\mathbf{3}_1^{(1)}$, $\mathbf{3}_1^{(2)}$, generators are a, b, c, and for $\mathbf{3}_2^{(1)}$, $\mathbf{3}_2^{(2)}$, generators are a, b, c'. Their character table are presented in table 2.

B Kronecker products

We display Kronecker products and calculation of Clebsch Gordan coefficients. The Kronecker products can be calculate from the character table in the previous subsection.

$$1_i \times 1_i = 1_1 \ (i = 1, 2), \qquad\qquad\qquad 1_1 \times 1_2 = 1_2 \times 1_1 = 1_2, \quad \text{(B.1)}$$

$$1_i \times 2_r = 2_r, \qquad\qquad\qquad\qquad 1_i \times 3_j^{(l)} = 3_{((i+j) \bmod 2)+1}^{(l)} \ (j, l = 1, 2), \quad \text{(B.2)}$$

$$2_r \times 2_r = 1_1 + 1_2 + 2_r \qquad\qquad\qquad\qquad\qquad\qquad\qquad (r = 1, 2, 3, 4), \quad \text{(B.3)}$$

$$2_a \times 2_b = 2_c + 2_d \qquad\qquad (a, b, c, d = 1, 2, 3, 4, \text{ different each other}), \quad \text{(B.4)}$$

$$2_r \times 3_j^{(l)} = 3_1^{(l)} + 3_2^{(l)}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.5)}$$

$$3_j^{(l)} \times 3_j^{(l)} = 3_1^{(l')} + 3_1^{(l')} + 3_2^{(l')} \qquad\qquad\qquad\qquad (l' = 1, 2, \ l \neq l'), \quad \text{(B.6)}$$

$$3_j^{(l)} \times 3_{j'}^{(l)} = 3_2^{(l')} + 3_2^{(l')} + 3_1^{(l')} \qquad\qquad (j' = 1, 2, \ j \neq j'), \qquad \text{(B.7)}$$

$$3_j^{(l)} \times 3_j^{(l')} = 1_1 + 2_1 + 2_2 + 2_3 + 2_4, \qquad\qquad\qquad\qquad\qquad\qquad \text{(B.8)}$$

$$3_j^{(l)} \times 3_{j'}^{(l')} = 1_2 + 2_1 + 2_2 + 2_3 + 2_4. \qquad\qquad\qquad\qquad\qquad\qquad \text{(B.9)}$$

C Multiplication of $\Delta(54)$

We present the relevant multiplication rules of $\Delta(54)$. The multiplication rules of two dimensional representation are given as

$$(x_1, x_2)_{2_r} \times (y_1, y_2)_{2_r} = (x_1 y_2 + x_2 y_1)_{1_1} + (x_1 y_2 - x_2 y_1)_{1_2} + (x_2 y_2, x_1 y_1)_{2_r} \ (r = 1, 2, 3, 4)$$

$$(x_1, x_2)_{2_1} \times (y_1, y_2)_{2_2} = (x_2 y_2, x_1 y_1)_{2_3} + (x_2 y_1, x_1 y_2)_{2_4}$$

$$(x_1, x_2)_{2_1} \times (y_1, y_2)_{2_3} = (x_2 y_2, x_1 y_1)_{2_2} + (x_2 y_1, x_1 y_2)_{2_4},$$

$$(x_1, x_2)_{2_1} \times (y_1, y_2)_{2_4} = (x_1 y_2, x_2 y_1)_{2_2} + (x_1 y_1, x_2 y_2)_{2_3},$$

$$(x_1, x_2)_{2_2} \times (y_1, y_2)_{2_3} = (x_2 y_2, x_1 y_1)_{2_1} + (x_1 y_2, x_2 y_1)_{2_4},$$

$$(x_1, x_2)_{2_2} \times (y_1, y_2)_{2_4} = (x_1 y_1, x_2 y_2)_{2_1} + (x_1 y_2, x_2 y_1)_{2_3},$$

$$(x_1, x_2)_{2_3} \times (y_1, y_2)_{2_4} = (x_1 y_2, x_2 y_1)_{2_1} + (x_1 y_1, x_2 y_2)_{2_2}.$$

The multiplication rules of three dimensional representation is given as

$$
\begin{aligned}
(x_1, x_2, x_3)_{3_1^{(1)}} \times (y_1, y_2, y_3)_{3_1^{(1)}} = {} & (x_1 y_1, x_2 y_2, x_3 y_3)_{3_1^{(2)}} \\
& + (x_2 y_3 + x_3 y_2, x_3 y_1 + x_1 y_3, x_1 y_2 + x_2 y_1)_{3_1^{(2)}} \\
& + (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)_{3_2^{(2)}},
\end{aligned}
$$

$$
\begin{aligned}
(x_1, x_2, x_3)_{3_1^{(2)}} \times (y_1, y_2, y_3)_{3_1^{(2)}} = {} & (x_1 y_1, x_2 y_2, x_3 y_3)_{3_1^{(1)}} \\
& + (x_2 y_3 + x_3 y_2, x_3 y_1 + x_1 y_3, x_1 y_2 + x_2 y_1)_{3_1^{(1)}} \\
& + (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)_{3_2^{(1)}},
\end{aligned}
$$

$$
\begin{aligned}
(x_1, x_2, x_3)_{3_1^{(1)}} \times (y_1, y_2, y_3)_{3_1^{(2)}} = {} & (x_1 y_1 + x_2 y_2 + x_3 y_3)_{1_1} \\
& + (x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3, \omega x_1 y_1 + \omega^2 x_2 y_2 + x_3 y_3)_{2_1} \\
& + (x_1 y_2 + \omega^2 x_2 y_3 + \omega x_3 y_1, \omega x_1 y_3 + \omega^2 x_2 y_1 + x_3 y_2)_{2_2} \\
& + (x_1 y_3 + \omega^2 x_2 y_1 + \omega x_3 y_2, \omega x_1 y_2 + \omega^2 x_2 y_3 + x_3 y_1)_{2_3} \\
& + (x_1 y_3 + x_2 y_1 + x_3 y_2, x_1 y_2 + x_2 y_3 + x_3 y_1)_{2_4},
\end{aligned}
$$

$$
\begin{aligned}
(x_1, x_2, x_3)_{3_1^{(2)}} \times (y_1, y_2, y_3)_{3_1^{(1)}} = {} & (x_1 y_1 + x_2 y_2 + x_3 y_3)_{1_1} \\
& + (x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3, \omega x_1 y_1 + \omega^2 x_2 y_2 + x_3 y_3)_{2_1} \\
& + (x_1 y_3 + \omega^2 x_2 y_1 + \omega x_3 y_2, \omega x_1 y_2 + \omega^2 x_2 y_3 + x_3 y_1)_{2_2} \\
& + (x_1 y_2 + \omega^2 x_2 y_3 + \omega x_3 y_1, \omega x_1 y_3 + \omega^2 x_2 y_1 + x_3 y_2)_{2_3} \\
& + (x_1 y_2 + x_2 y_3 + x_3 y_1, x_1 y_3 + x_2 y_1 + x_3 y_2)_{2_4},
\end{aligned}
$$

$$\begin{aligned}
(x_1, x_2, x_3)_{3_2^{(1)}} \times (y_1, y_2, y_3)_{3_2^{(1)}} &= (x_1 y_1, x_2 y_2, x_3 y_3)_{3_1^{(2)}} \\
&\quad + (x_2 y_3 + x_3 y_2, x_3 y_1 + x_1 y_3, x_1 y_2 + x_2 y_1)_{3_1^{(2)}} \\
&\quad + (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)_{3_2^{(2)}}, \\
(x_1, x_2, x_3)_{3_2^{(2)}} \times (y_1, y_2, y_3)_{3_2^{(2)}} &= (x_1 y_1, x_2 y_2, x_3 y_3)_{3_1^{(1)}} \\
&\quad + (x_2 y_3 + x_3 y_2, x_3 y_1 + x_1 y_3, x_1 y_2 + x_2 y_1)_{3_1^{(1)}} \\
&\quad + (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)_{3_2^{(1)}}, \\
(x_1, x_2, x_3)_{3_1^{(1)}} \times (y_1, y_2, y_3)_{3_2^{(2)}} &= (x_1 y_1 + x_2 y_2 + x_3 y_3)_{1_2} \\
&\quad + (x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3, -\omega x_1 y_1 - \omega^2 x_2 y_2 - x_3 y_3)_{2_1} \\
&\quad + (x_1 y_2 + \omega^2 x_2 y_3 + \omega x_3 y_1, -\omega x_1 y_3 - \omega^2 x_2 y_1 - x_3 y_2)_{2_2} \\
&\quad + (x_1 y_3 + \omega^2 x_2 y_1 + \omega x_3 y_2, -\omega x_1 y_2 - \omega^2 x_2 y_3 - x_3 y_1)_{2_3} \\
&\quad + (x_1 y_3 + x_2 y_1 + x_3 y_2, -x_1 y_2 - x_2 y_3 - x_3 y_1)_{2_4}, \\
(x_1, x_2, x_3)_{3_1^{(2)}} \times (y_1, y_2, y_3)_{3_2^{(1)}} &= (x_1 y_1 + x_2 y_2 + x_3 y_3)_{1_2} \\
&\quad + (x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3, -\omega x_1 y_1 - \omega^2 x_2 y_2 - x_3 y_3)_{2_1} \\
&\quad + (x_1 y_3 + \omega^2 x_2 y_1 + \omega x_3 y_2, -\omega x_1 y_2 - \omega^2 x_2 y_3 - x_3 y_1)_{2_2} \\
&\quad + (x_1 y_2 + \omega^2 x_2 y_3 + \omega x_3 y_1, -\omega x_1 y_3 - \omega^2 x_2 y_1 - x_3 y_2)_{2_3} \\
&\quad + (x_1 y_2 + x_2 y_3 + x_3 y_1, -x_1 y_3 - x_2 y_1 - x_3 y_2)_{2_4}.
\end{aligned}$$

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